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Elastic-plastic constitutive equation accounting for microstructure

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Abstract. In this study, we focus on the modeling of solid structures that include microstructures observed in particle-dispersed composites. The finite element modeling can be used to clarify how the macroscopic behaviors of solid structures are influenced by the microstructures. In such a case, if the whole structure including the microstructures is modeled by the finite elements, an enormous number of finite elements and enormous amount of computational time are required. To overcome such difficulties, we propose a new method for modeling microstructures. In this method, an explicit form of the stress-strain relation covering both elastic and elastic-plastic regions is derived from the equivalent inclusion method proposed by Eshelby that provides mathematical solutions for stress and strain at an arbitrary point inside and outside the inclusion. The derived elastic-plastic constitutive equation takes account of the microstructures, so that the effect of microstructures on the macroscopic behaviors can be obtained from the conventional finite element method by using such a constitutive equation without modeling microstructures in the finite element analysis. The effectiveness of the proposed constitutive equation is verified for a simple problem by comparing the results of the one-element finite element analyses using the proposed constitutive equation with those of the detailed finite element analyses using multi-element finite element modeling.

Introduction

Composite materials have inhomogeneity in the viewpoint of microstructure, and the inhomogeneity affects mechanical properties of composites. The finite element method can be used to clarify how the macroscopic behaviors of solid structures are influenced by the microstructures. In such a case, if the whole structure including the microstructure is modeled by the finite elements, an enormous number of finite elements and enormous amount of computational time are required. To overcome such difficulties, various studies have been performed on the macroscopic constitutive equation for particle-dispersed composites in order to predict their mechanical behaviors. Among them, the Eshelby's equivalent inclusion method [1] has been used for predicting mechanical behaviors of particle-dispersed composites. For example, Mori and Tanaka [2] developed the mean field theory based on the Eshelby's equivalent inclusion method. They assumed that stress and strain are uniform in each phase of a composite, and derived the elastic constitutive equation of the composite. On the other hand, Tandon and Weng [3] extended the Mori-Tanaka's theory to an elastic-plastic constitutive equation for a particle-dispersed composite. In their theory, however, stress and strain distributions are not taken into account in deriving the elastic-plastic constitutive equation. In actual particle-dispersed composites, there exist stress and strain distributions around particles, which may affect the mechanical behaviors of composites. In the present study, we derive an elastic-plastic constitutive equation that represents the macroscopic behaviors of a particle-dispersed composite by taking account of stress and strain distributions in the matrix. In present theory, we modify the Eshelby's equivalent inclusion method to calculate elastic-plastic stress and strain fields in the matrix. The effectiveness of the proposed constitutive equation is verified for a simple problem by comparing the results of the one-element finite element analyses using the proposed constitutive equation with those of the detailed finite element analyses using multi-element finite element modeling.

Eshelby's theory

We assume that a single particle or inclusion exists in uniform infinite matrix. According to the Eshelby's equivalent inclusion method, we can evaluate stresses and strains in the inclusion and matrix under the uniform loading, assuming that the real inclusion is replaced by the virtual inclusion or the equivalent inclusion with the same material of the matrix and a certain arbitrary eigenstrain. The total strain of the equivalent inclusion is given as follows:

$$\varepsilon_2 = \varepsilon_2^e + \varepsilon^* \tag{1}$$

where ε^* is the eigenstrain, and the subscripts 0, 1 and 2 indicate the matrix at the infinite location, the matrix and the inclusion, respectively. We assume that a composite is subjected to a uniform strain ε_0 at the infinite location. Here, we define the stress difference between ε_1 and ε_0 as ε_1^c, and that between ε_2 and ε_0 as ε_2^c, respectively.

$$\varepsilon_1(x) = \varepsilon_0 + \varepsilon_1^c(x), \qquad \varepsilon_2 = \varepsilon_0 + \varepsilon_2^c \tag{2}$$

In Eq.(2), $\varepsilon_1(x)$ and $\varepsilon_1^c(x)$ indicate the functions of position. On the other hand, ε_2 and ε_2^c in the inclusion are assumed to be constant. We can obtain $\varepsilon_1^c(x)$ and ε_2^c from the Eshelby tensors $(S_{out}(x), S_{in})$ and eigenstrain.

$$\varepsilon_1^c(x) = S_{out}(x) : \varepsilon^*, \qquad \varepsilon_2^c = S_{in} : \varepsilon^* \tag{3}$$

Although the eigenstrain given by Eq.(3) is arbitrary, it has to satisfy the equivalence, as shown in Fig. 1. So we consider equivalent condition for the stress between the real inclusion and equivalent inclusion, and we obtain the following equation.

$$\sigma_2 = \sigma_{eqv} = D_1^e : (\varepsilon_2 - \varepsilon^*), \qquad \sigma_2 = D_2^e : \varepsilon_2 \tag{4}$$

where D_1^e and D_2^e are the elastic matrix for the matrix material and that for the inclusion,

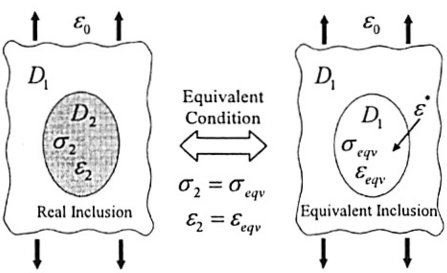


Fig. 1. Concept of Equivalent inclusion method

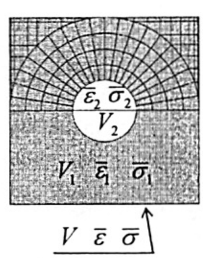


Fig. 2. A unit cell in the present model

respectively. Using Eqs.(2)-(4), we obtain the eigenstrain as a function of ε_0 that satisfies the equivalent condition.

$$\varepsilon^{*} = A_0 : \varepsilon_0 \tag{5}$$

$$A_0 = \left[\left(I \otimes I - (D_1^e)^{-1} : D_2^e\right)^{-1} - S_{in}\right]^{-1} \tag{6}$$

where I denotes the unit matrix. By substituting Eq.(5) into Eq.(3) and using Eq.(2), we can obtain the strains in the matrix and the inclusion, respectively, as follows:

$$\varepsilon_1(x) = \left(I \otimes I + S_{out}(x) : A_0\right) : \varepsilon_0, \tag{7}$$

$$\varepsilon_2 = \left(I \otimes I + S_{in} : A_0\right) : \varepsilon_0. \tag{8}$$

As shown in Eqs.(7) and (8), both the strains in the matrix and the inclusion are calculated from the strain at the infinite location ε_0 and the several material properties included in D_1^e and D_2^e. Then the stresses in the matrix and the inclusion are written as

$$\sigma_1(x) = D_1^e : \left(I \otimes I + S_{out}(x) : A_0\right) : \varepsilon_0 \tag{9}$$

$$\sigma_2 = D_2^e : \left(I \otimes I + S_{in} : A_0\right) : \varepsilon_0 \tag{10}$$

Macroscopic constitutive law

In this section, we derive a macroscopic constitutive equation from the equations shown in the previous section. Fig. 2 shows a unit cell in the present model consisting of a lot of background cells for numerical integration, which will be mentioned later. For the unit cell, we define the average strain of the matrix $\bar{\varepsilon}_1$ and that of the inclusion $\bar{\varepsilon}_2$ as follows:

$$\bar{\varepsilon}_1 = \frac{1}{V_1}\int_{V_1} \varepsilon_1 dV, \quad \bar{\varepsilon}_2 = \frac{1}{V_2}\int_{V_2} \varepsilon_2 dV, \tag{11}$$

where V_1, V_2 and V are the volume of matrix, that of inclusion and the overall volume, respectively. We assume that the average strain of the overall volume $\bar{\varepsilon}$ is given as:

$$\bar{\varepsilon} = (1-f)\bar{\varepsilon}_1 + f\bar{\varepsilon}_2, \tag{12}$$

where f is the volume fraction. By substituting Eqs.(7), (8) and (11) into Eq.(12), $\bar{\varepsilon}$ is written as a function of ε_0 as follows:

$$\bar{\varepsilon} = \alpha : \varepsilon_0, \tag{13}$$

$$\alpha = \frac{1}{V}\left[\int_{V_1} I \otimes I + S_{out}(x) : A_0 dV + \int_{V_2} I \otimes I + S_{in} : A_0 dV\right]. \tag{14}$$

Similarly, the average stress in the matrix $\bar{\sigma}_1$, the average stress in the inclusion $\bar{\sigma}_2$ and the average stress of the overall volume $\bar{\sigma}$ are written as:

$$\bar{\sigma}_1 = \frac{1}{V_1}\int_{V_1} \sigma_1 dV, \quad \bar{\sigma}_2 = \frac{1}{V_2}\int_{V_2} \sigma_2 dV, \tag{15}$$

$$\overline{\sigma} = \beta : \varepsilon_0, \tag{16}$$

$$\beta = \frac{1}{V}\left[\int_{V_1} D_1^e : (I \otimes I + D(x) : A_0) dV + \int_{V_2} D_2^e : (I \otimes I + S : A_0) dV\right]. \tag{17}$$

The average stress $\overline{\sigma}$ is written as a function of ε_0 as well as the average strain $\overline{\varepsilon}$. α and β are 4th order tensors that relate the average strain and the average stress with the strain at the infinite location. Finally, we can obtain the relationship between the average stress and the average strain by eliminating ε_0 from Eq.(13) and Eq.(16) as follows:

$$\overline{\sigma} = \overline{D}^e : \overline{\varepsilon} = \beta : \alpha^{-1} : \overline{\varepsilon}. \tag{18}$$

We can regard the above equation as a constitutive equation for a particle-dispersed composite. When the matrix material is in a plastic state and the inclusion remains elastic, a constitutive equation for an elastic-plastic problem can be obtained by changing the elastic matrix of the matrix material D_1^e in Eqs.(6) and (17) to the elastic-plastic matrix of the matrix material D_1^{ep} and revising Eq.(18) to an incremental form. Conclusively, a constitutive equation for an elastic-plastic problem of a particle-dispersed composite is given as follows:

$$d\overline{\sigma} = \overline{D}^{ep} : d\overline{\varepsilon} = \beta : \alpha^{-1} : d\overline{\varepsilon}, \tag{19}$$

$$\alpha = \frac{1}{V}\left[\int_{V_1} I \otimes I + S_{out}(x) : A_0 dV + \int_{V_2} I \otimes I + S_{in} : A_0 dV\right], \tag{20}$$

$$\beta = \frac{1}{V}\left[\int_{V_1} D_1^{ep} : (I \otimes I + D(x) : A_0) dV + \int_{V_2} D_2^e : (I \otimes I + S : A_0) dV\right], \tag{21}$$

where

$$A_0 = \left[\left(I \otimes I - (D_1^{ep})^{-1} : D_2^e\right)^{-1} - S_{in}\right]^{1}. \tag{22}$$

Numerical integration is required to calculate α and β both for an elastic problem and for an elastic-plastic problem. Fig. 2 also shows the background cells for the numerical integration. Due to the symmetry, the numerical integration is performed for a half region. When the matrix material is in a plastic state and the inclusion remains elastic, the incremental strain and incremental stress in the matrix material are given as follows, by replacing D_1^e in Eqs.(6) and (9) with :

$$d\varepsilon_1(x) = \left(I \otimes I + S_{out}(x) : A_0\right) : d\varepsilon_0, \tag{23}$$

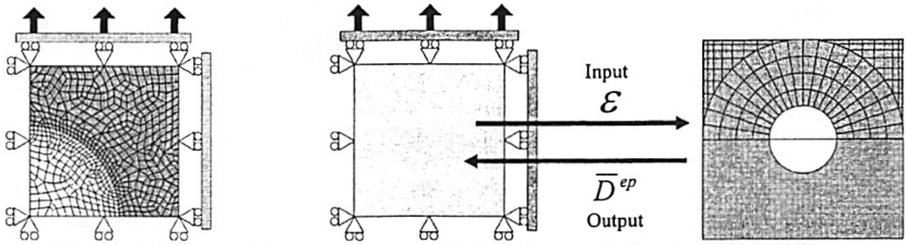
$$d\sigma_1(x) = D_1^{ep} : \left(I \otimes I + S_{out}(x) : A_0\right) : d\varepsilon_0. \tag{24}$$

Results and discussion

We performed the analyses to verify the accuracy of the constitutive equation derived in the present study. The problem analyzed here is a particle-dispersed composite under uniaxial loading. The results of the analyses using the constitutive equation are compared with those of the analyses using detailed finite element modeling. Fig. 3(a) shows the detailed finite element modeling consisting of a lot of finite elements, and each element has the material properties of the matrix material or those of the inclusion according as it belongs to the matrix region or the inclusion region. On the other hand, if we employ the constitutive equation given in the present study, we can use the one-element finite element modeling as shown in Fig. 3(b) without considering the detailed structure.

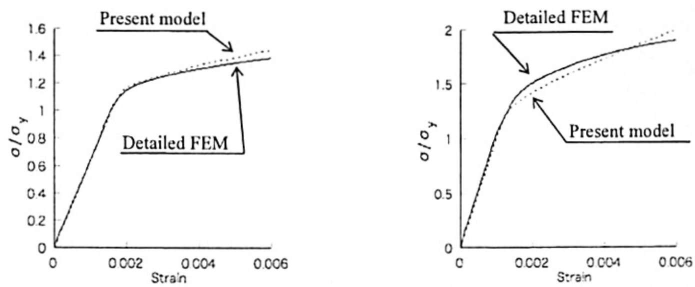
Table 1 . Material parameters

	E[GPa]	σ_y[MPa]	ν	B[MPa]	n
Matrix	30.0	60.0	0.33	135.02	7.79
Inclusion	130	300.0	0.33	332.88	7.79



(a) Detailed finite element model (b) One-element finite element modeling

Fig. 3. Finite element modeling



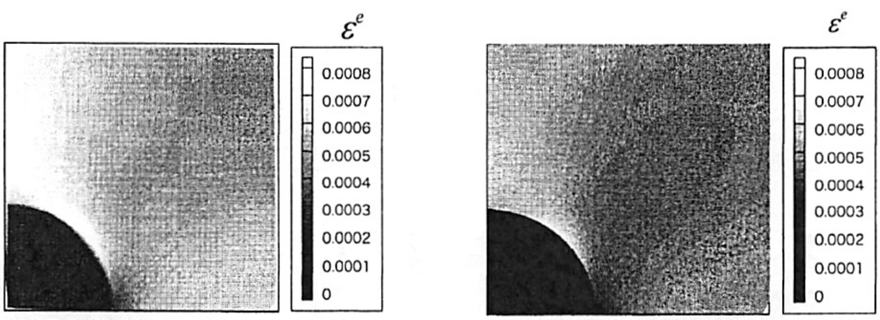
(a) Volume fraction 10% (b) Volume fraction 50%

Fig. 4. Comparison between present model and detailed finite element model

After obtaining the solution of the one-element finite element modeling, we can calculate the strain and stress at an arbitrary point x in the matrix material using Eqs.(23) and (24). The material properties used in the present study are given in Table 1, where E, σ_y and ν denote Young's modulus, yield stress and Poisson's ratio, respectively, and B and n are the coefficients of Ramberg-Osgood relation given by

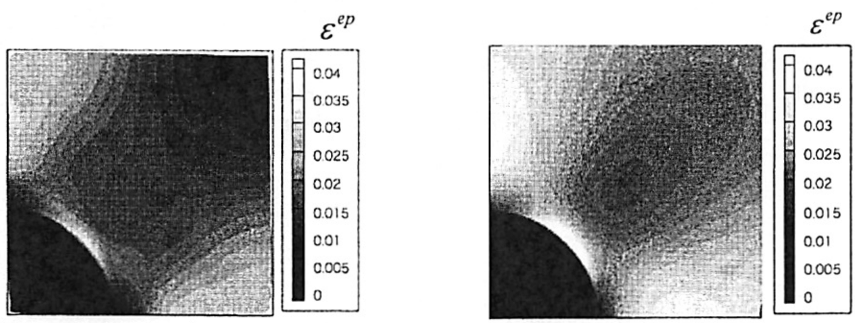
$$\varepsilon = (\sigma / E) + (\sigma / B)^n . \tag{25}$$

The results of the stress-strain relation at the loading location are shown in Fig. 4 for the volume fractions of the inclusion f of 10% and 50%. The result of the one-element finite element modeling using the proposed constitutive equation is compared with that of the detailed finite element modeling. Both results agree well not only in the elastic region but also in the elastic-plastic region. Figs. 5 and 6 compare the distribution of the equivalent strain ε^{eqv} obtained from the one-element finite element modeling using the proposed constitutive equation with that of the detailed finite element modeling for the volume fraction of the inclusion f of 10%. In Fig. 5, the strain at the loading location is 0.05%, and the whole structure remains elastic. On the other hand, in Fig. 6, the strain at the loading location is 1.5%, and a part of the matrix region becomes plastic. For the elastic case shown in Fig. 5, the result of present model is in good agreement with the result of detailed finite element modeling, especially in the strain distribution near the inclusion surface



(a) Detailed finite element modeling (b) Present model

Fig. 5. Distribution of equivalent strain for an elastic case



(a) Detailed finite element modeling (b) Present model

Fig. 6. Distribution of equivalent strain for an elastic-plastic case

and in the distribution of the lower strain area spreading along the 45-degree. For the elastic-plastic case shown in Fig. 6, the comparison is not so good in comparison with the elastic case, but the present model captures the features of the strain distributions characterized by the highest strain near the inclusion surface and the lower strain area spreading along the 45-degree, although the lower strain area is not so clear, compared with the detailed finite element modeling.

Concluding remarks

In the present study, we derived an elastic-plastic constitutive equation that expresses the macroscopic behaviors of a particle-dispersed composite by taking account of stress and strain distributions in the matrix. The effectiveness of the proposed constitutive equation is verified by comparing the results of the one-element finite element analyses using the proposed constitutive law with those of the detailed finite element analyses using multi-element finite element modeling. The proposed constitutive equation enables the stress analysis of sold structures made of composite materials without modeling microstructures of composite materials.

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